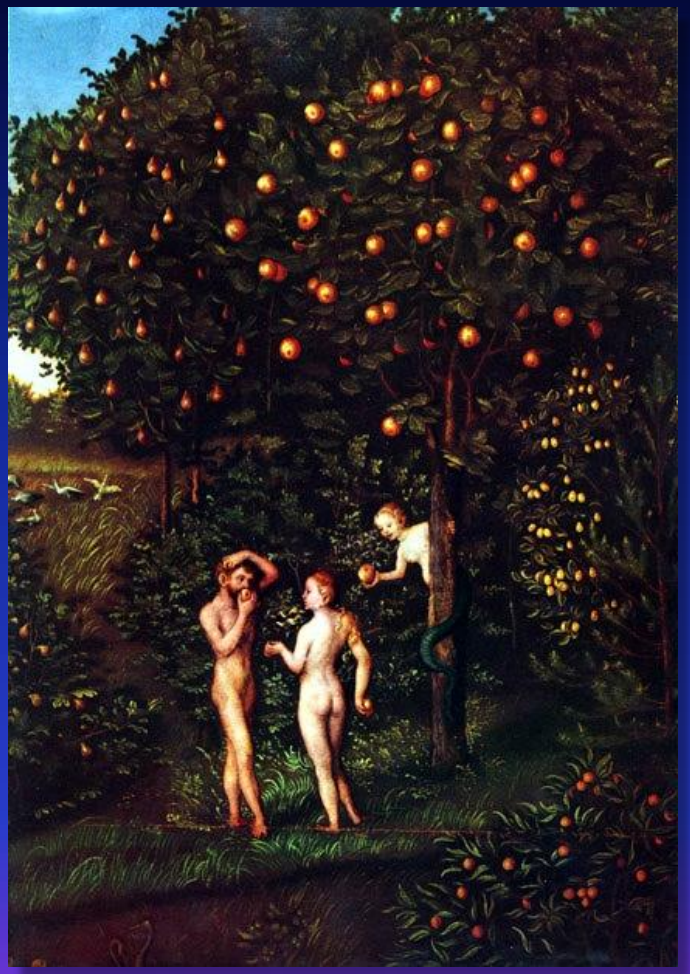
Statistics: The Central Limit Theorem

- ✓ Click "slide show" to start this presentation as a show.
- ✓ Remember: focus & think about each point; do not just passively click.



Cranach, *Tree of Knowledge* [of Good and Evil] (1472)

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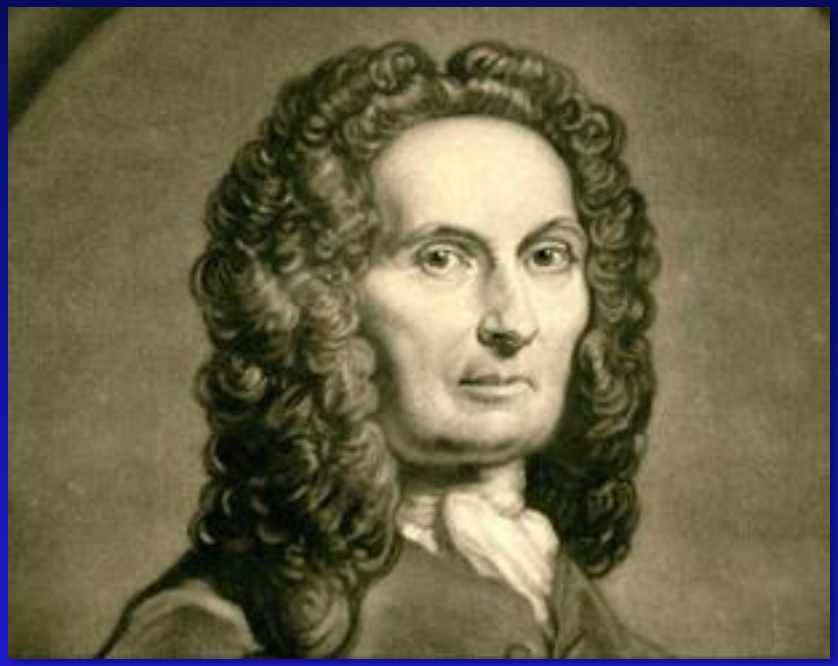
The central limit theorem and sampling distributions

	Alpha Levels				
df	0.10	0.05	0.02	0.01	0.001
8	1.860	2.306	2.896	3.355	5.041
9	1.833	2.262	2.821	3.250	4.781
10	1.812	2.228	2.764	3.169	4.587
11	1.796	2.201	2.718	3.106	4.437
12	1.782	2.179	2.681	3.055	4.318
13	1.771	2.160	2.650	3.012	4.221
14	1.761	2.145	2.624	2.977	4.140
15	1.753	2.131	2.602	2.947	4.073
18	1.734	2.101	2.552	2.878	3.922
20	1.725	2.086	2.528	2.845	3.850
25	1.708	2.060	2.485	2.787	3.725
30	1.697	2.042	2.457	2.750	3.646
40	1.684	2.021	2.423	2.704	3.551
60	1.671	2.000	2.390	2.660	3.460
120	1.658	1.980	2.358	2.617	3.373
∞	1.645	1.960	2.326	2.576	3.291



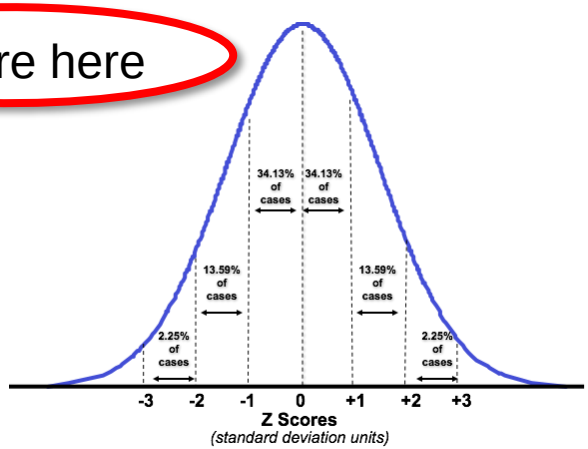
Abraham De Moivre . (2012). Famous-Mathematicians.com. Retrieved October 8, 2015 from http://www.famous-mathematicians.com/abraham-de-moivre/

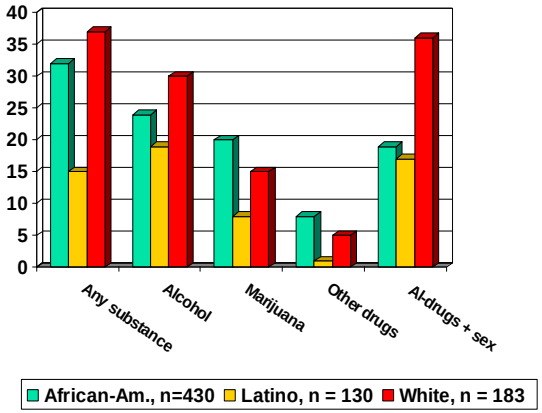
Abraham de Moivre, French Hugenot refugee in London, originator of the Central Limit Theorem.

Click for a short biography.

1. Introduction to statistics & number scales
2. The Z score and the normal distribution
3. The logic of research; Plato's Allegory of the Cave
4. Testing hypotheses: The critical ratio
5. Calculating a t score
6. Testing t: The Central Limit Theorem
7. Correlations: Measures of association

You are here





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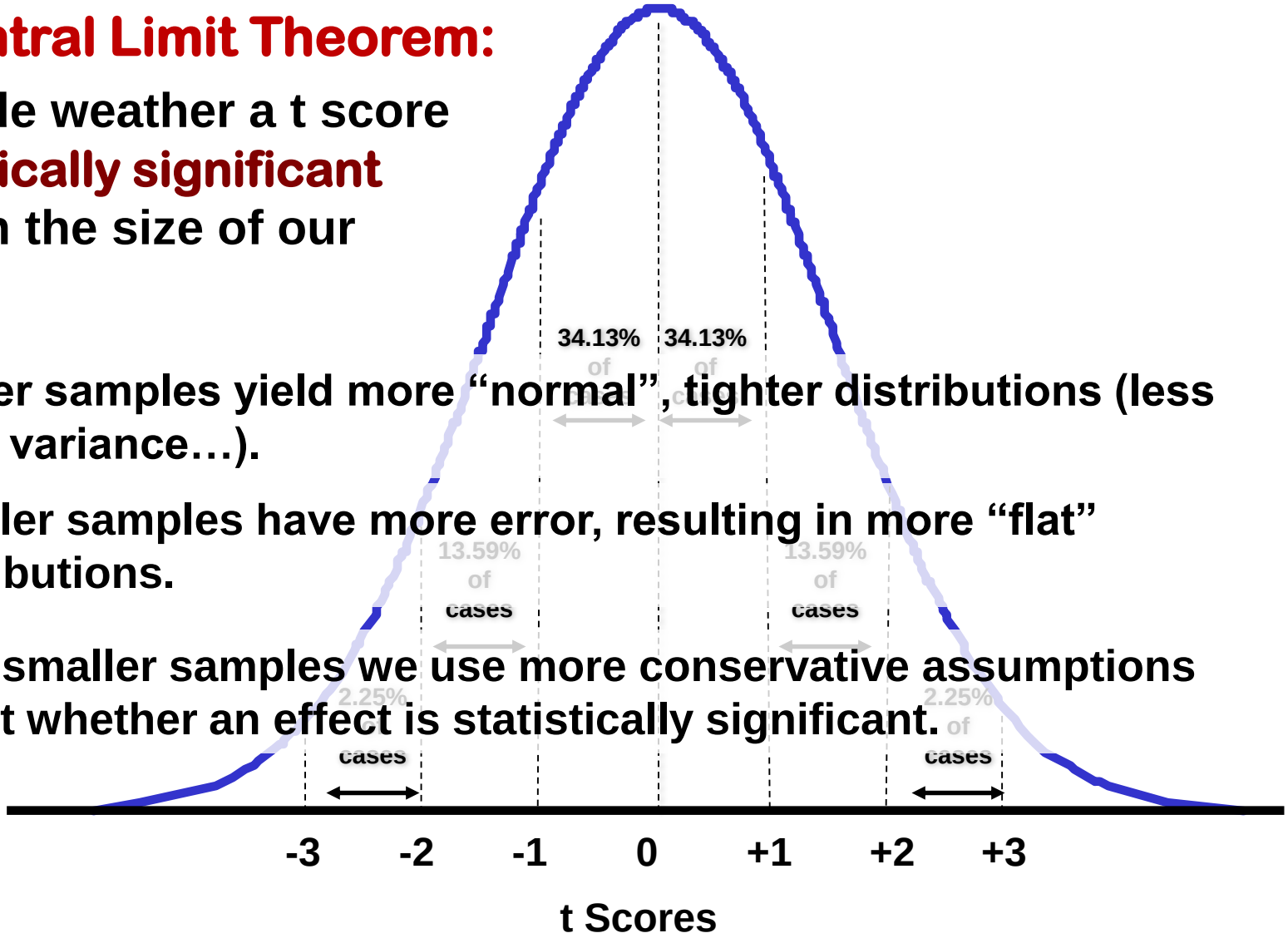
The Central Limit Theorem:

We decide weather a t score is statistically significant based on the size of our sample:

- **Larger samples yield more "normal", tighter distributions (less error variance…).**
- **Smaller samples have more error, resulting in more "flat" distributions.**
- **With smaller samples we use more conservative assumptions about whether an effect is statistically significant.**

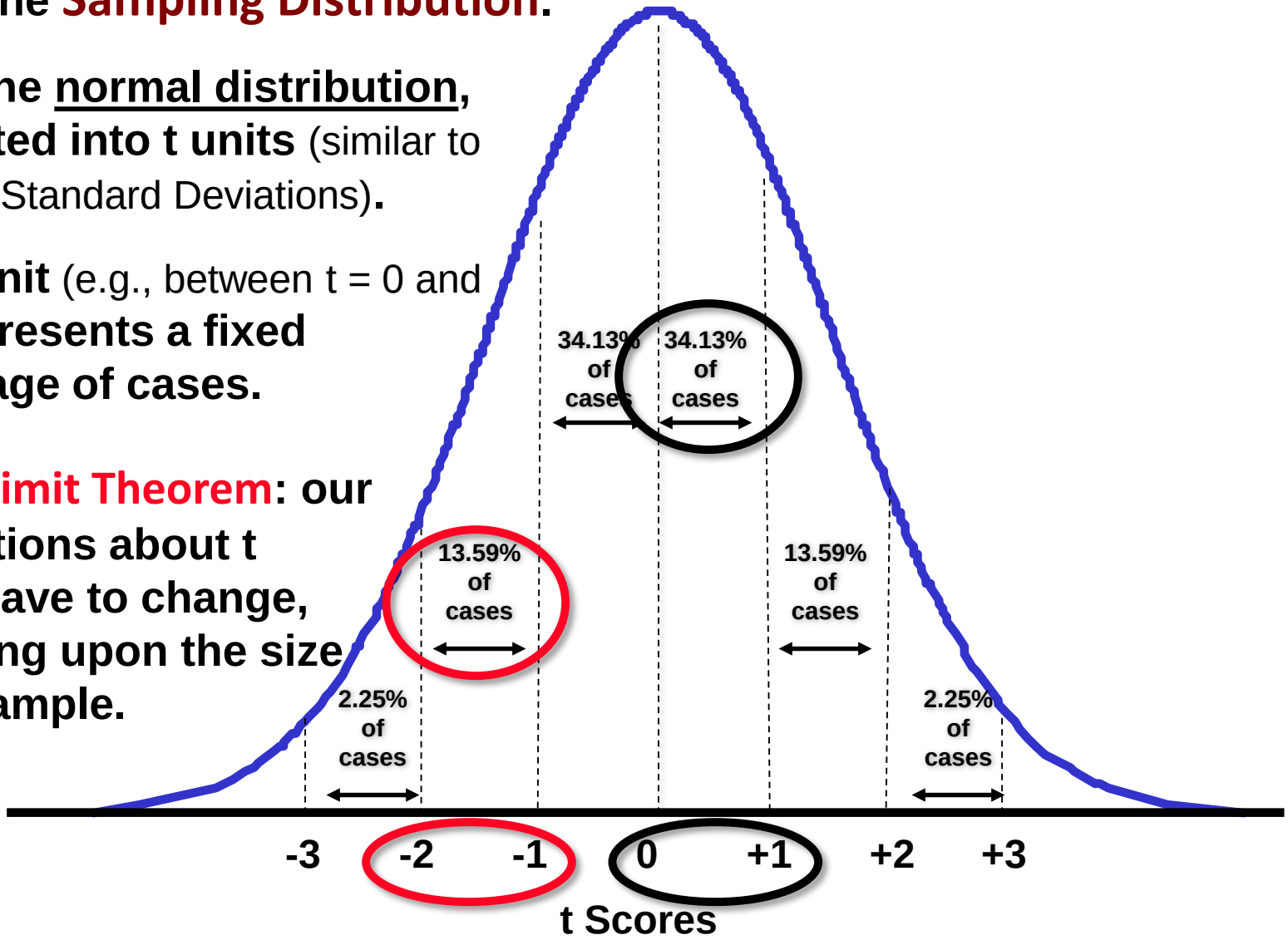


Here is the Sampling Distribution.

This is the normal distribution, segmented into t units (similar to Z units or Standard Deviations)**.**

Each t unit (e.g., between t = 0 and t = 1) **represents a fixed percentage of cases.**

Central Limit Theorem: our assumptions about t values have to change, depending upon the size of our sample.



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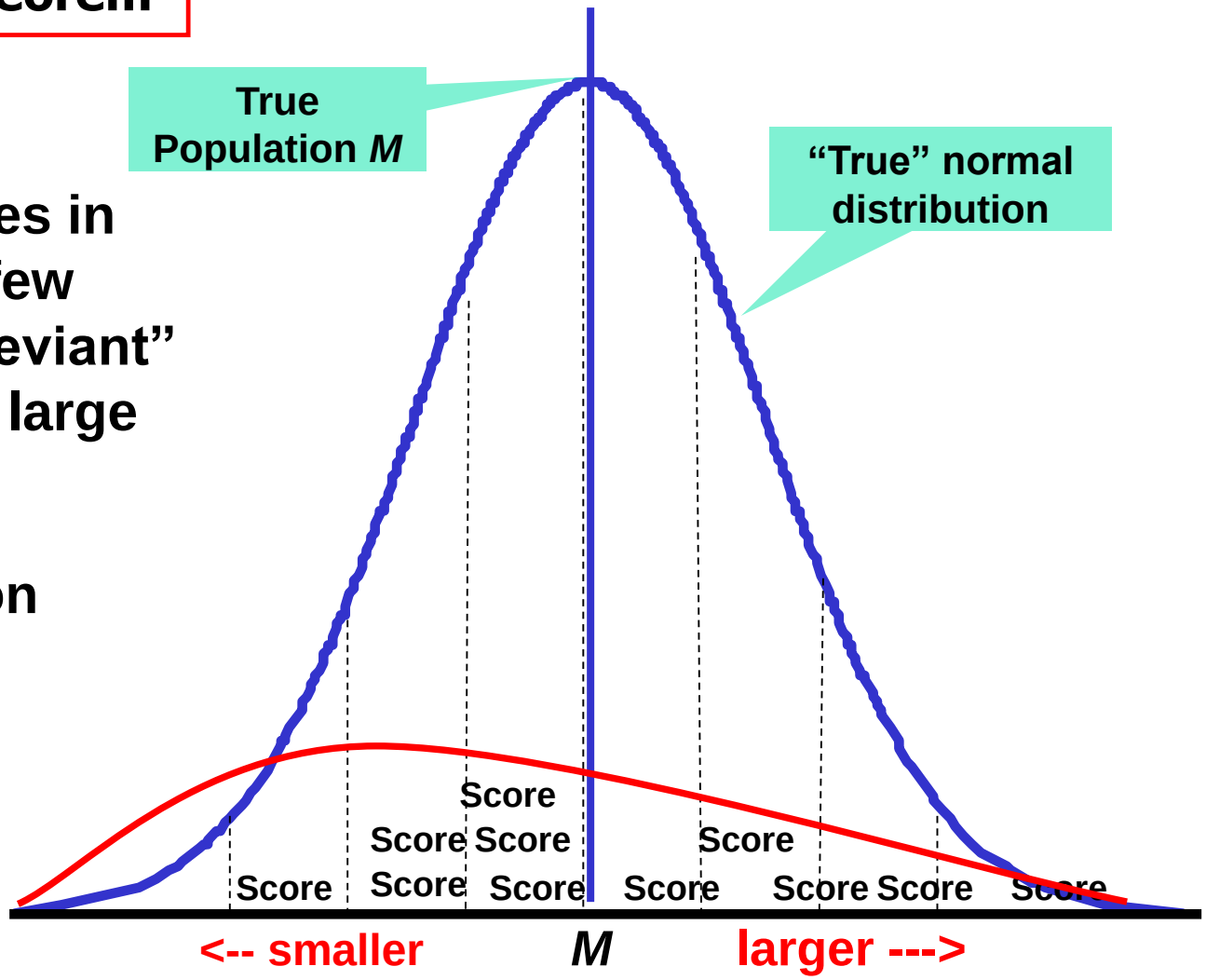
The Central Limit Theorem; small samples

Central Limit Theorem

- ✓ With few scores in the sample a few extreme or "deviant" values have a large effect.
- ✓ The distribution is "flat" or has high variance.



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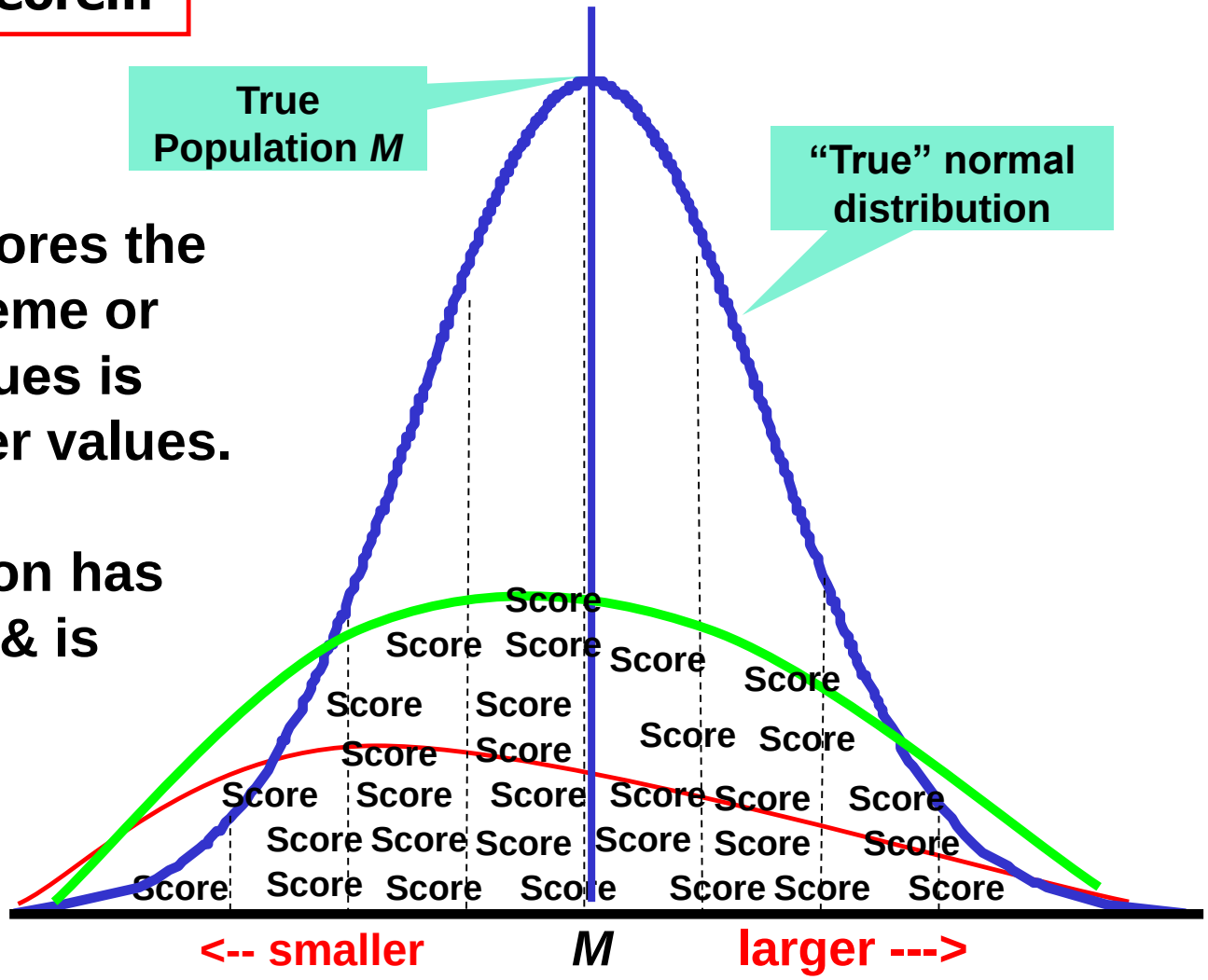
Central Limit Theorem

- ✓ With more scores the effect of extreme or "deviant" values is offset by other values.
- ✓ The distribution has less variance & is more normal.



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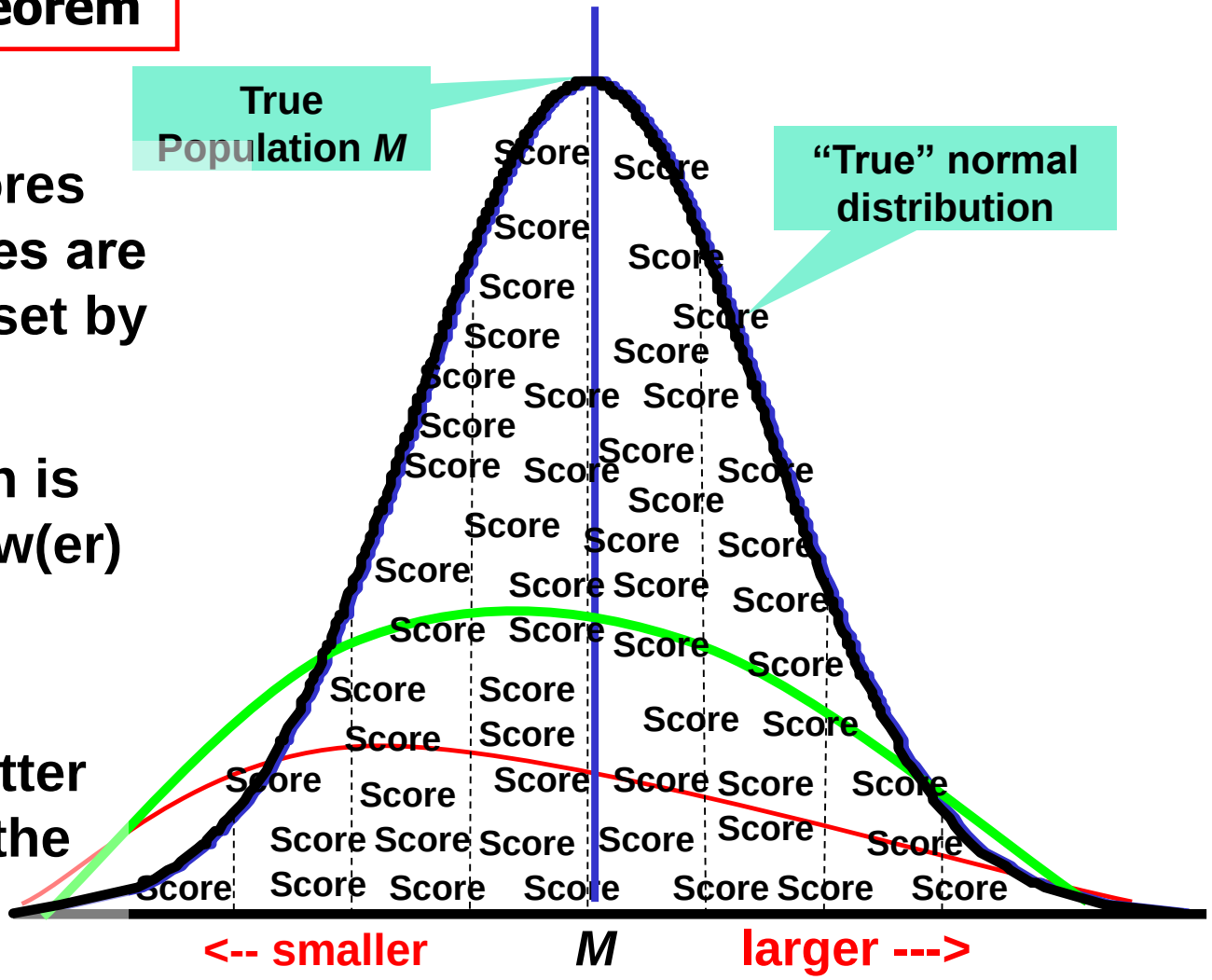
Central Limit Theorem

- With many scores "deviant" values are completely offset by other values.
- The distribution is normal, with low(er) variance.
- The sampling distribution better approximates the population distribution



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Sample Size and the Normal Distribution: Pascal's quincunx.

How do larger samples produce more "normal" distributions?

Imagine an array of equally spaced pins.

Balls fall through the array with a 50% (0.5) chance of bouncing left or right.

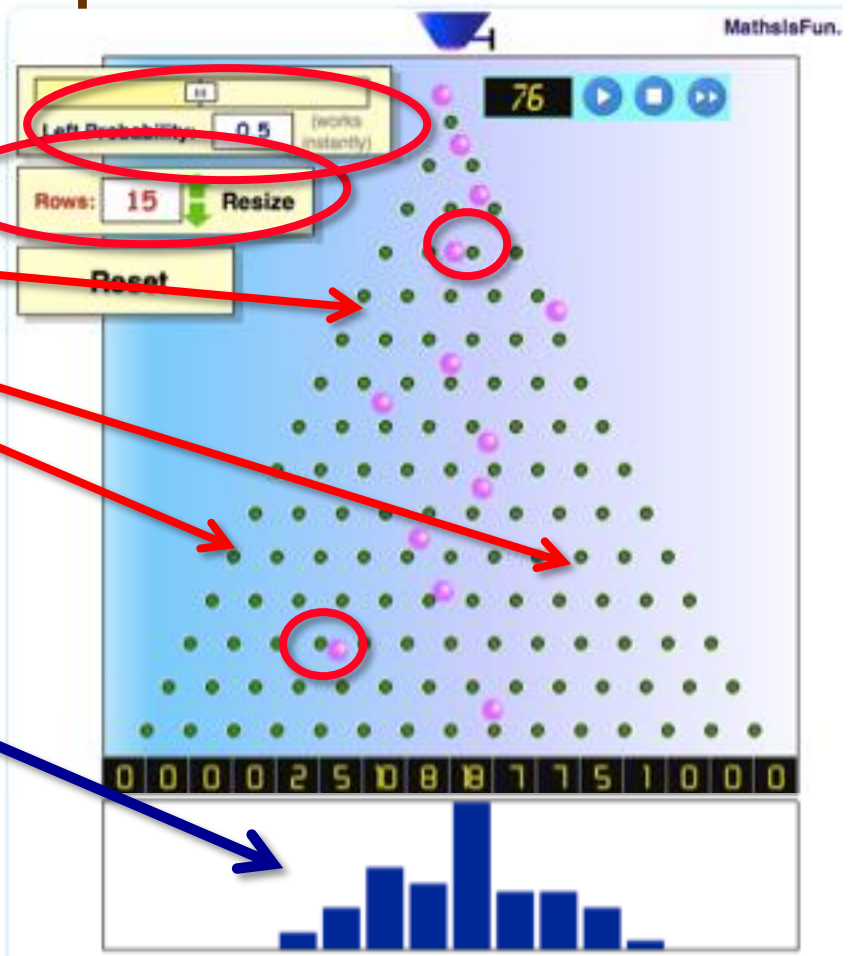
This makes the path of the balls completely random.

The balls collect in bins, below

This forms our frequency distribution.

This array is set for 15 rows; more rows allow for more error.

This is set for 50::50 (0.5); changing this biases the distribution.



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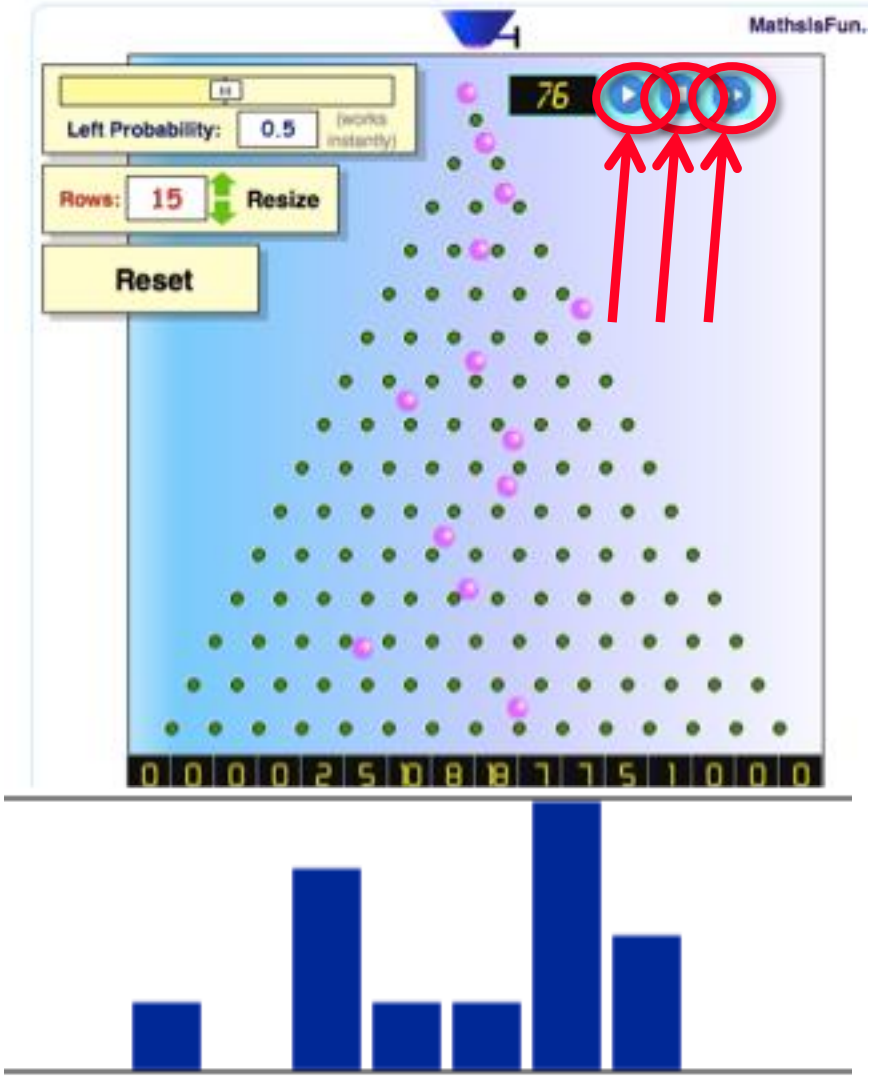
Sample Size and the Normal Distribution: Pascal's quincunx.

You can click the image at the end to see how random data forms different distributions.

1. **After you begin, stop the demonstration after a dozen balls** (a small sample).
2. **Note how "flat" and non-normal it is.**
3. **Restart: after more balls accumulate watch how more "normal" the distribution becomes.**
4. **Put the demonstration into 'hyperdrive' to see what a very large sample looks like.**



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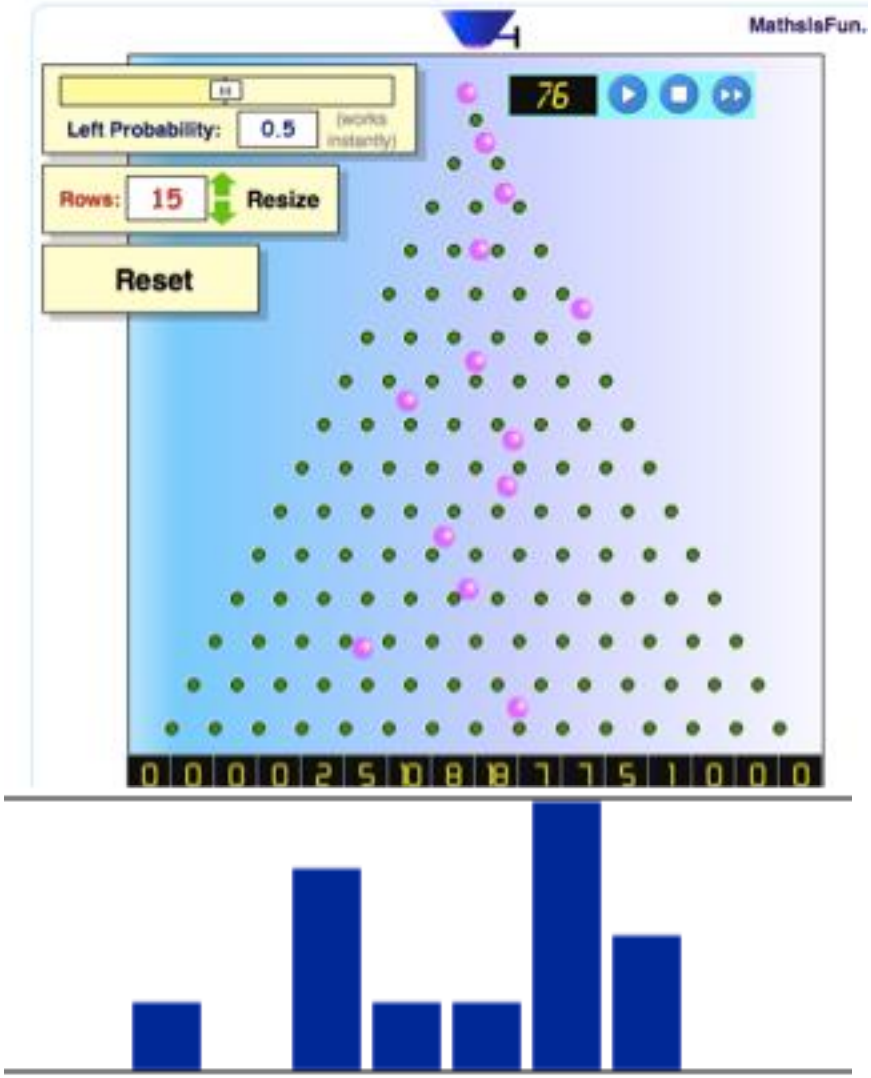
Now click the image to start.



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Central limit theorem & evaluating t scores

The same logic applies with *samples* we use to test hypotheses.

1. With smaller groups the *M* score for each group reflects more error variance.
2. This increases the likelihood that **error variance**, not an experimental effect, led to differences between Ms.
3. Since smaller samples (**lower *df***) have more variance, t must be larger for us to consider it **statistically significant** (< 5% likely to have occurred by chance alone).
4. We evaluate t vis-à-vis a **sampling distribution** based on the **df for the experiment.**
5. **Critical value** for *t* with *p* <.05 thus goes up or down depending upon sample size (*df*)

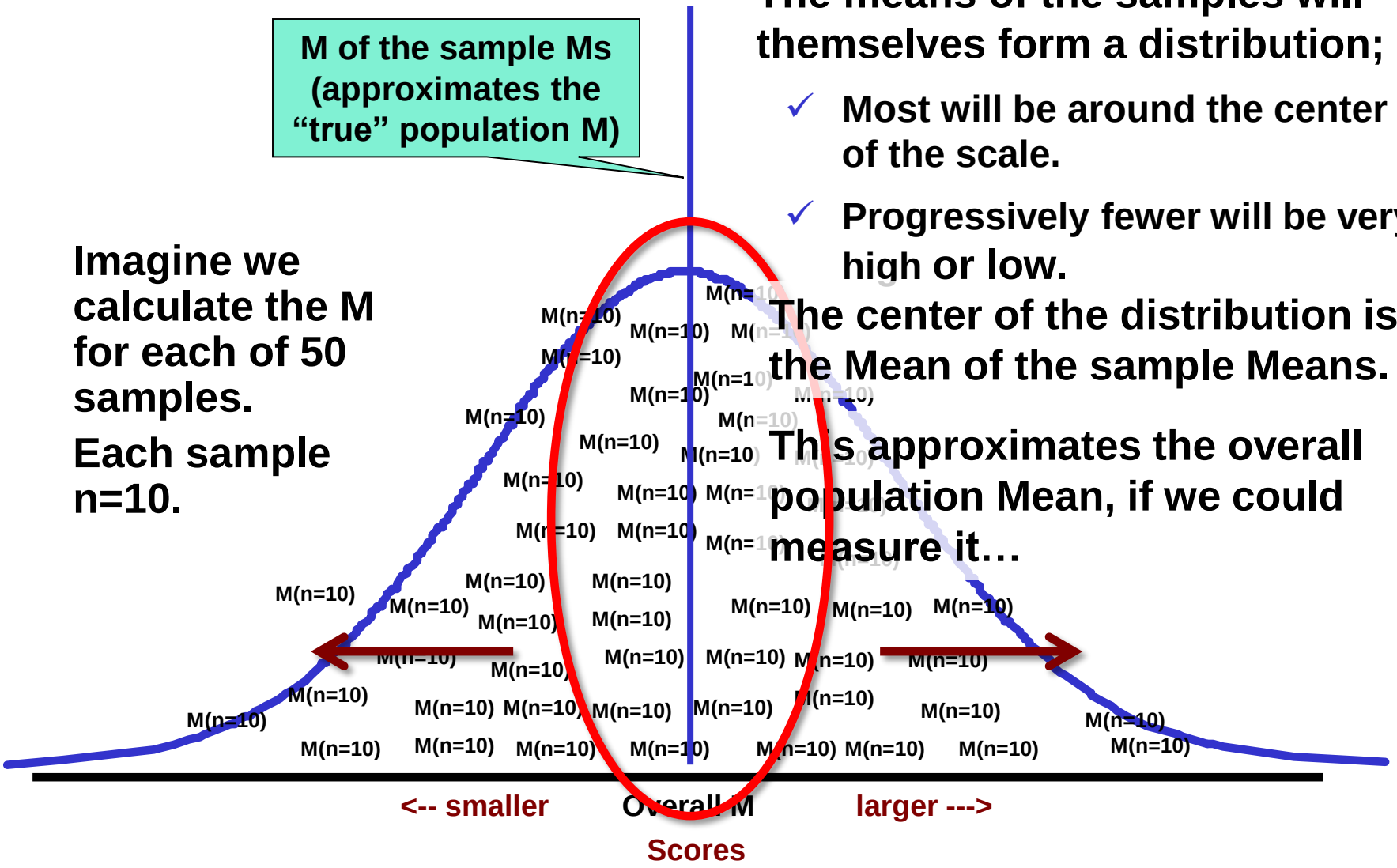
Dr. David McKirnan, davidmck@uic.edu

Central Limit Theorem applied to a sampling distribution: How well do small samples reflect the "true" population?

M of the sample Ms (approximates the "true" population M)

Imagine we calculate the M for each of 50 samples.

Each sample n=10.



The means of the samples will themselves form a distribution;

- ✓ Most will be around the center of the scale.
- ✓ Progressively fewer will be very high or low.

The center of the distribution is the Mean of the sample Means.

This approximates the overall population Mean, if we could measure it…

Dr. David McKirnan, davidmck@uic.edu

Central Limit Theorem applied to a sampling distribution:
How well do small samples reflect the "true" population?

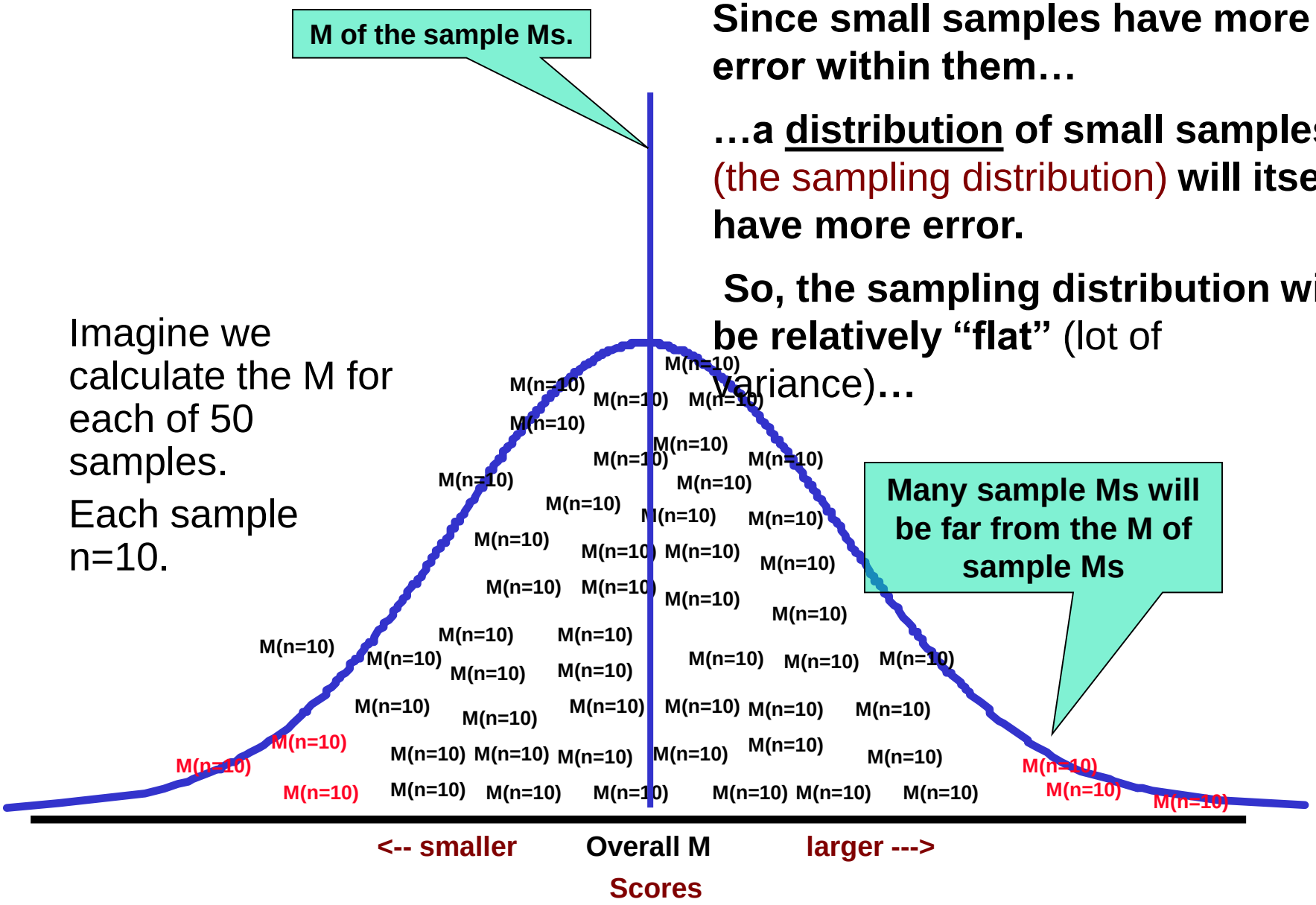
Since small samples have more error within them…

…a <u>distribution</u> of small samples (the sampling distribution) **will itself have more error.**

So, the sampling distribution will be relatively "flat" (lot of variance)**…**

Imagine we calculate the M for each of 50 samples.

Each sample n=10.



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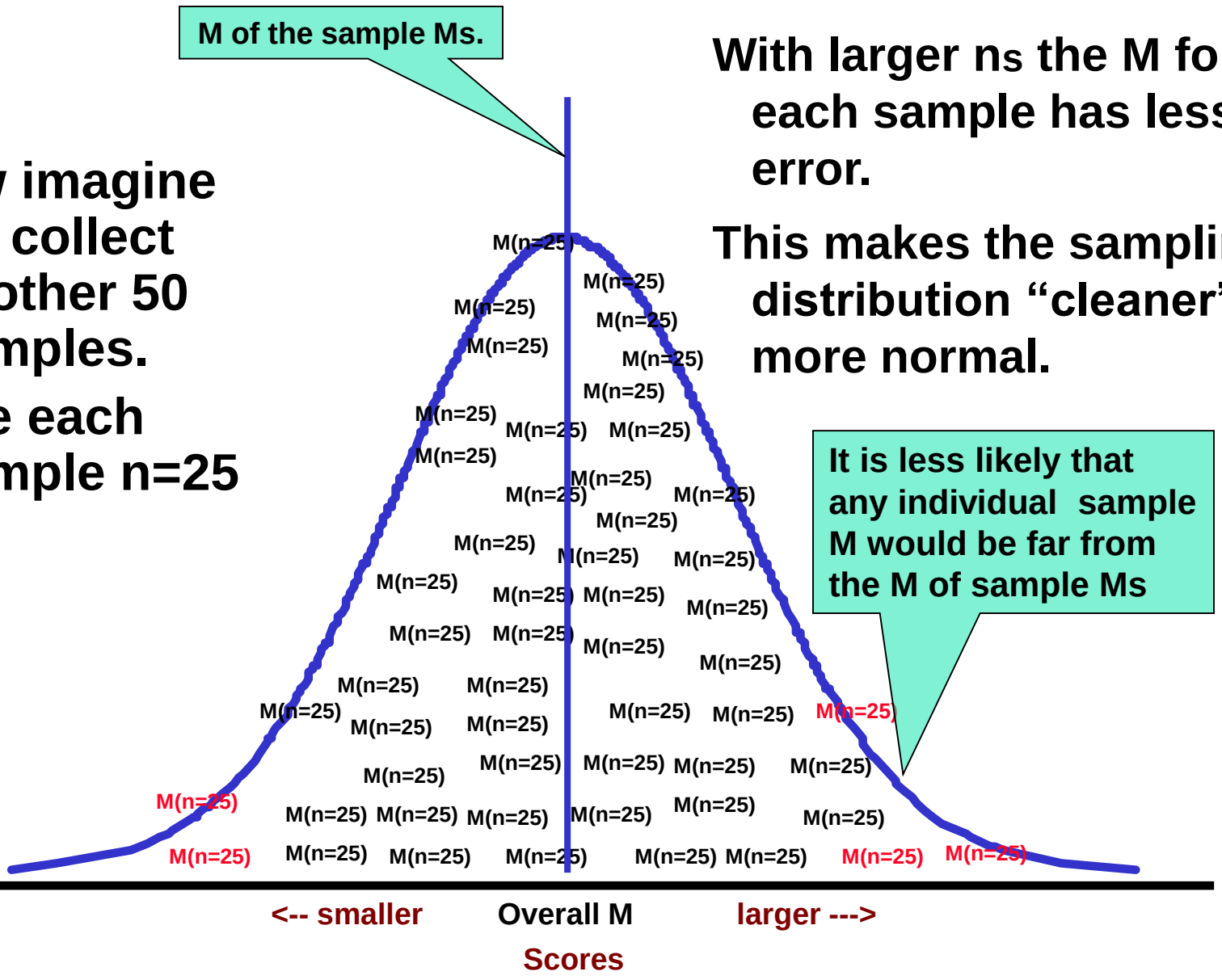
Central Limit Theorem applied to a sampling distribution.

Now imagine we collect another 50 samples.

Here each sample n=25



With larger ns the M for each sample has less error.

This makes the sampling distribution "cleaner" and more normal.

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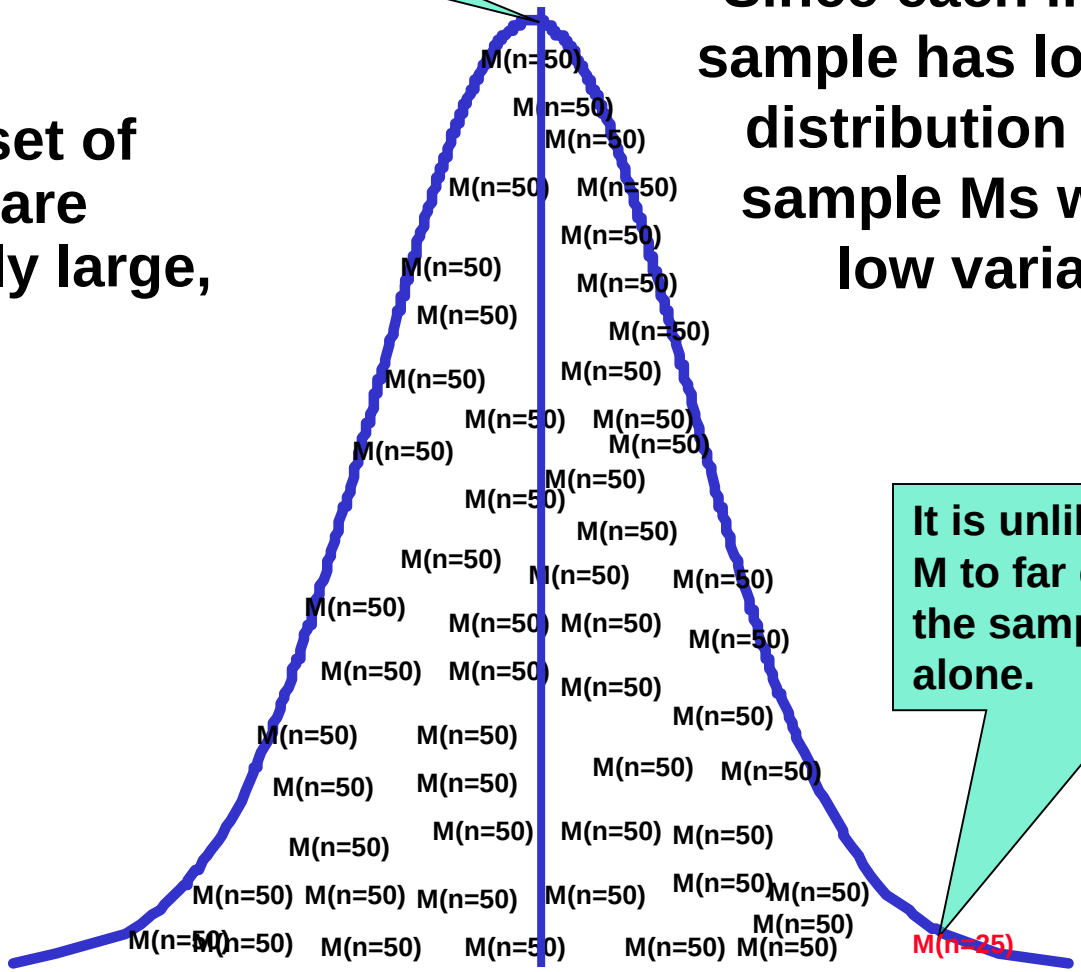
Central Limit Theorem applied to a sampling distribution.

M of the sample Ms.

Since each individual sample has low error, a distribution of large sample Ms will have low variance.

Our third set of samples are each fairly large, say n=50



It is unlikely for a sample M to far exceed the M of the sample Ms by chance alone.

<-- smaller **Overall M** **larger --->**

Scores

Dr. David McKirnan, davidmck@uic.edu

Central limit theorem: critical values

- **Central limit theorem:**
  - **When df ≥ 120 we assume a perfectly normal distribution.**
    (Here **Z = t**; we do not have to compensate for smaller samples.)
  - **With smaller samples, we assume more error in each group.**
    - **When df < 120 we use t to *estimate* a sampling distribution based on the dfs of the samples (i.e., *n*s of groups being sampled).**
- **Alpha [α]:**
  - **Probability criterion for "statistical significance," typically *p* < .05**
- **Critical value**
  - **Cut off point for alpha on the sampling distribution:**
  - **With df ≥ 120 critical value for p<.05 = ± 1.98 (Z = t)**
  - **With df < 120 we adjust the critical value based on the sampling distribution we use**
    - **As df goes down we assume a more conservative sampling distribution, and use a larger critical value for p <.05.**

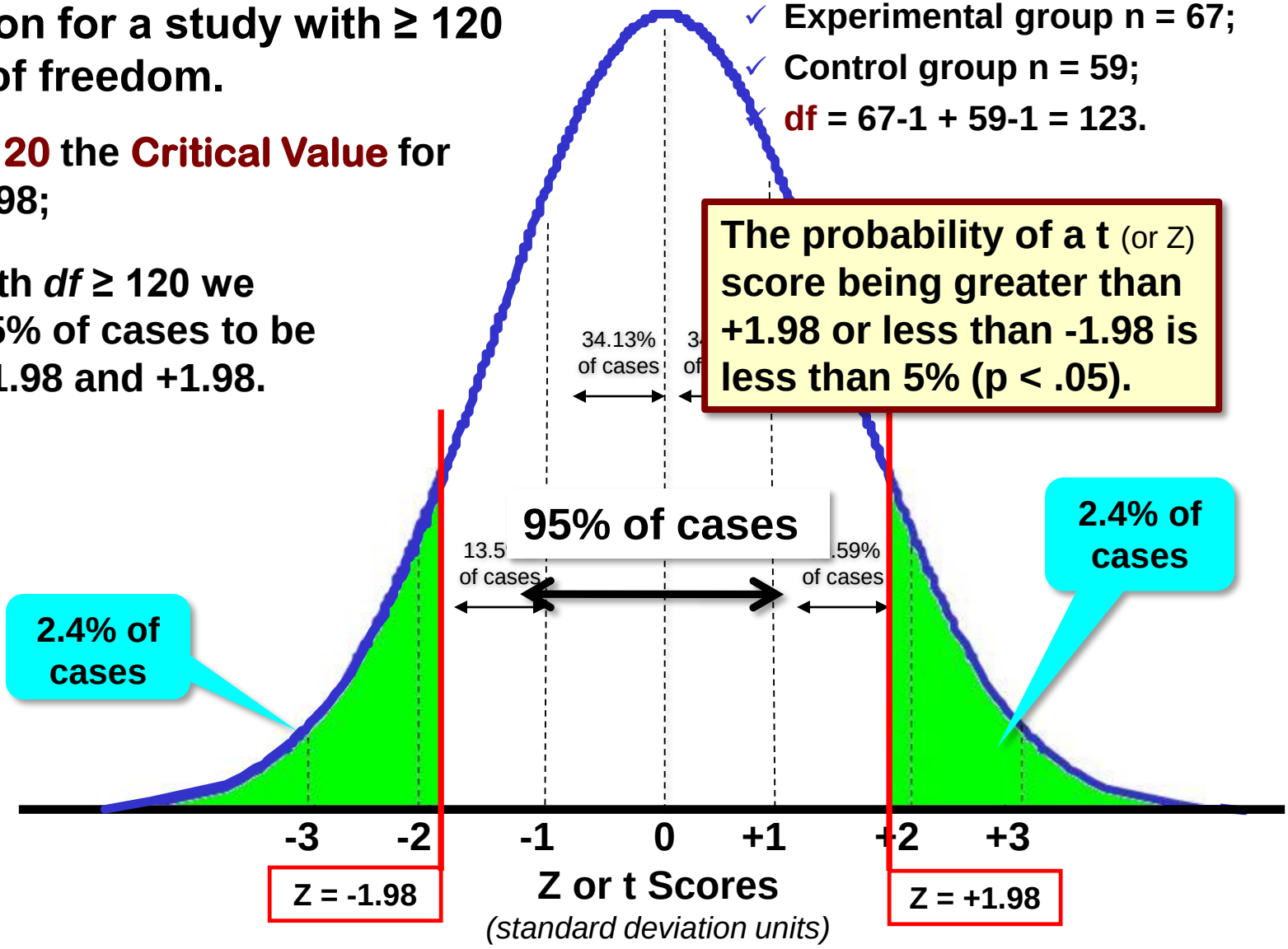
Sampling Distributions and Critical Values

- **Here is the estimated sampling distribution for a study with ≥ 120 degrees of freedom.**
- **With *df* ≥ 120 the Critical Value for p<.05 is 1.98;**
- **That is, with *df* ≥ 120 we assume 95% of cases to be between -1.98 and +1.98.**

For example:

- ✓ **Experimental group n = 67;**
- ✓ **Control group n = 59;**
- ✓ **df = 67-1 + 59-1 = 123.**

The probability of a t (or Z) **score being greater than +1.98 or less than -1.98 is less than 5% (p < .05).**



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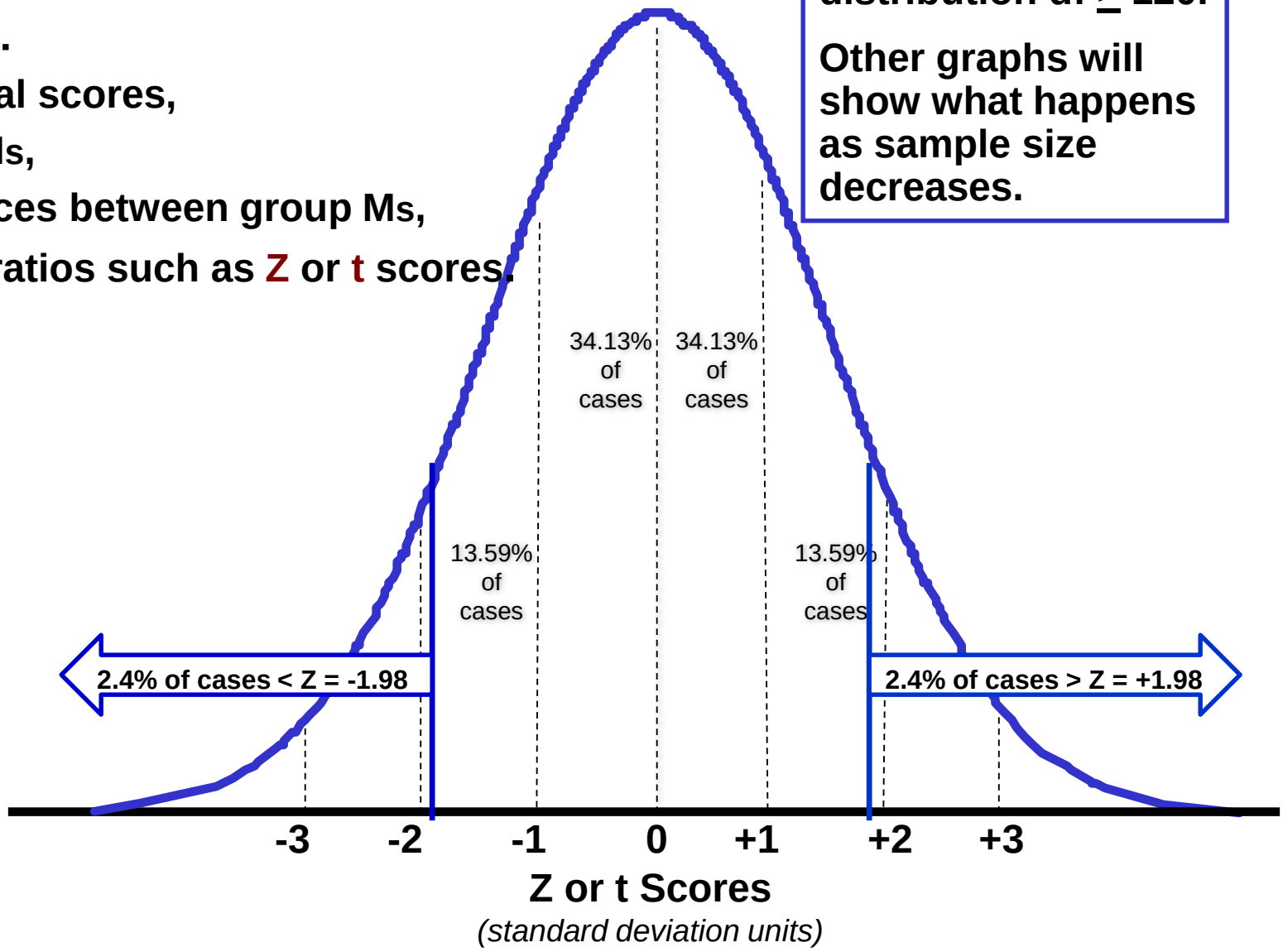
Sampling Distributions and Critical Values

- **It does not matter whether "cases" refers to…**
  - ✓ **Individual scores,**
  - ✓ **Group Ms,**
  - ✓ **Differences between group Ms,**
  - ✓ **Critical ratios such as Z or t scores.**

This sampling distribution df ≥ 120.

Other graphs will show what happens as sample size decreases.



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Now let's imagine small group sizes.

- ✓ With fewer *df* we **estimate** a more flat, "error-ful" curve.
- ✓ At *df* = 18 the critical value for p<.05 becomes larger, from 1.98 to 2.10,
- ✓ …a more conservative test.

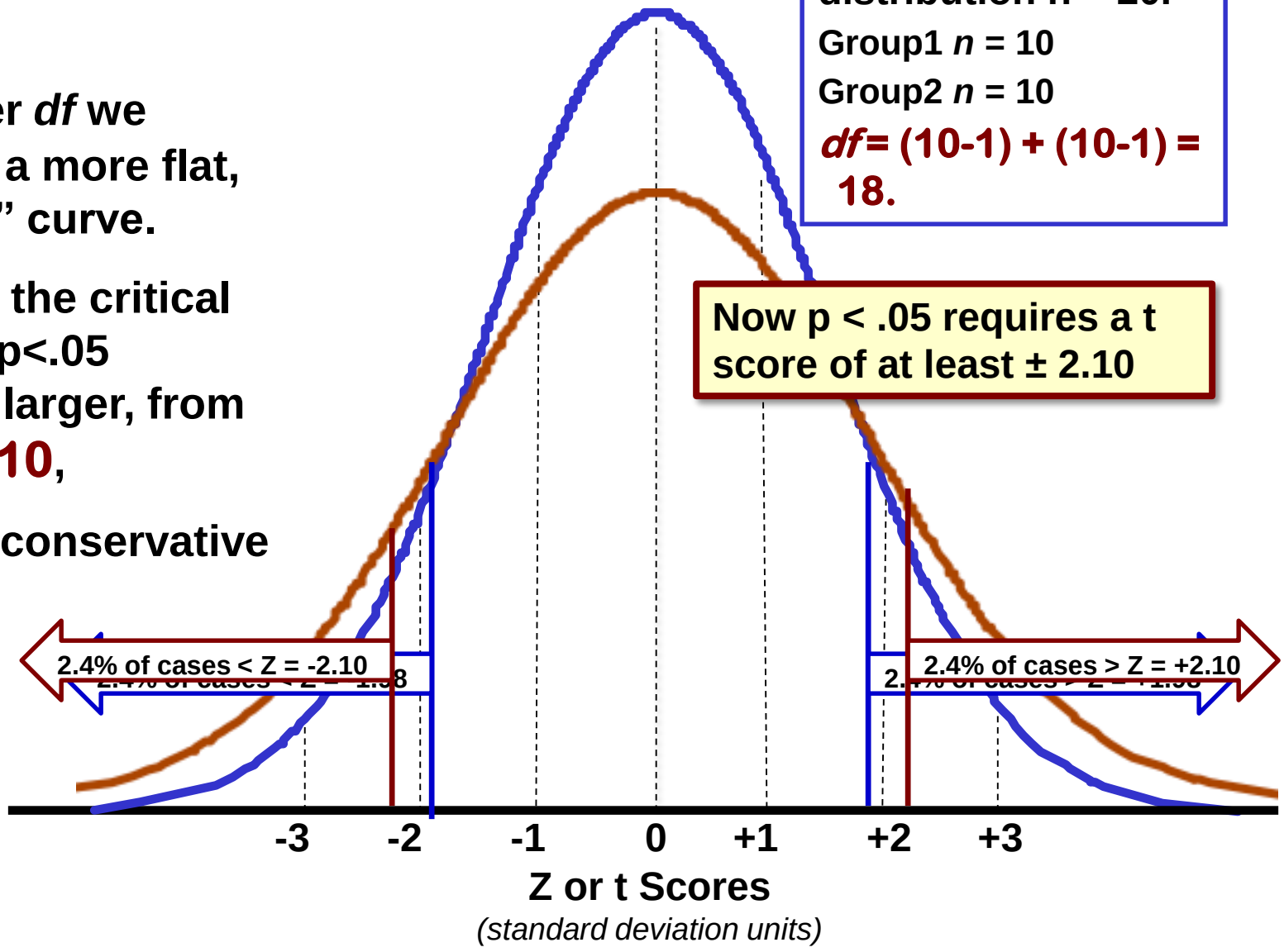
This sampling distribution n = 20.

Group1 *n* = 10

Group2 *n* = 10

***df*= (10-1) + (10-1) = 18.**

Now p < .05 requires a t score of at least ± 2.10



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Here is an example with a very small sample.

- ✓ With few *df* we estimate a more flat, "error-ful" curve.
- ✓ At *df* = 8 the critical value for p<.05 becomes larger, from 1.98 or 2.10 to 2.30.
- ✓ …a very conservative test.

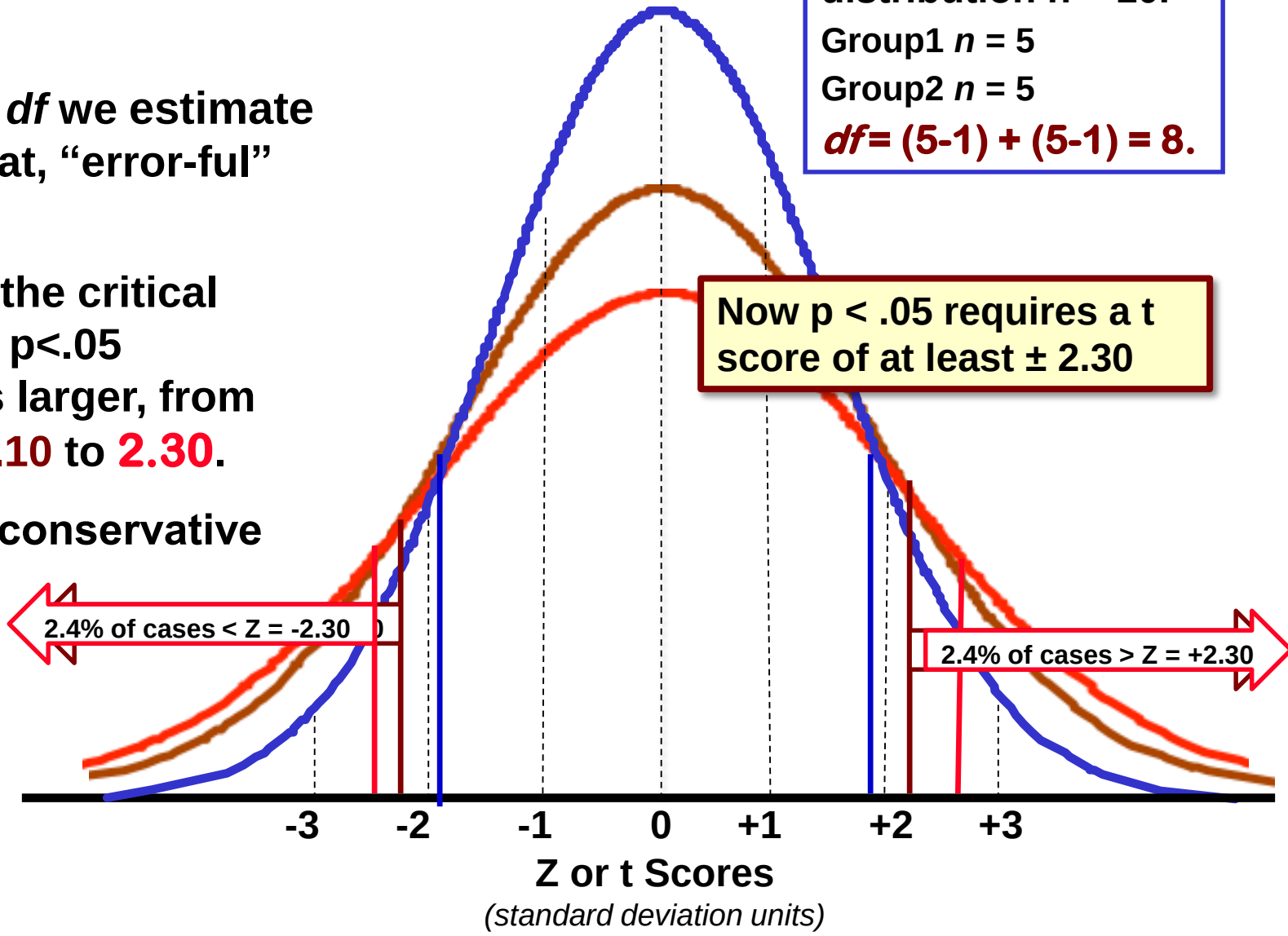
This sampling distribution n = 10.

Group1 *n* = 5

Group2 *n* = 5

***df*= (5-1) + (5-1) = 8.**

Now p < .05 requires a t score of at least ± 2.30



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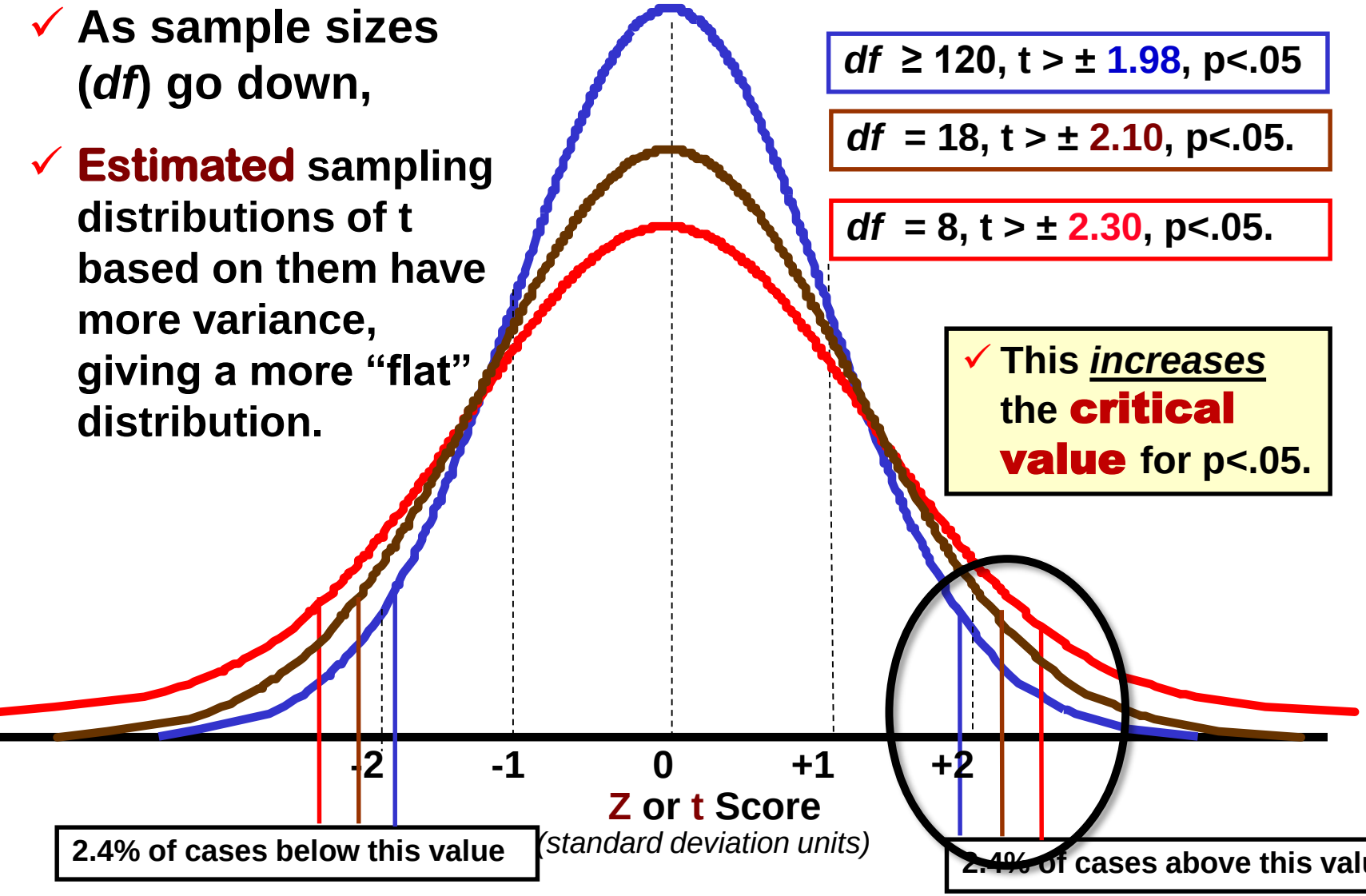
Central Limit Theorem; variations in sampling distributions

- ✓ As sample sizes (*df*) go down,
- ✓ **Estimated** sampling distributions of t based on them have more variance, giving a more "flat" distribution.


df ≥ 120, t > ± 1.98, p<.05
df = 18, t > ± 2.10, p<.05.
df = 8, t > ± 2.30, p<.05.
✓ This *increases* the **critical value** for p<.05.
-2
-1
0
+1
+2
Z or t Score
(standard deviation units)
2.4% of cases below this value
2.4% of cases above this value

Dr. David McKirnan, davidmck@uic.edu

A t-table contains:

	Alpha Levels				
df	**0.10**	**0.05**	**0.02**	**0.01**	**0.001**
8	1.860	**2.306**	2.896	3.355	5.041
9	1.833	**2.262**	2.821	3.250	4.781
10	1.812	**2.228**	2.764	3.169	4.587
11	1.796	**2.201**	2.718	3.106	4.437
12	1.782	**2.**		3.055	4.318
13	1.771	**2.**		3.012	4.221
14	1.761	**2.145**	2.624	2.977	4.140
15	1.753	**2.131**	2.602	2.947	4.073
20	1.725	**2.086**	2.528	2.845	3.850
25	1.708	**2.060**	2.485	2.787	3.725
30	1.697	**2.042**	2.457	2.750	3.646
40	1.684	**2.021**	2.423	2.704	3.551
60	1.671	**2.000**	2.390	2.660	3.460
120	1.658	**1.980**	2.358	2.617	3.373
∞	1.645	**1.960**	2.326	2.576	3.291

Critical values of t

- **Degrees of freedom (df)**
  Size of the research samples:
  $(n_{group1} - 1) + (n_{grp2} - 1)$
- **Alpha levels**
  % likelihood of a t occurring by chance.
- **Critical Values**
  Value t must exceed to be statistically significant [not occurring by chance] **at a given alpha.**

Dr. David McKirnan, davidmck@uic.edu

Critical values of t (2 tailed test)

df	Alpha Levels 0.10	0.05	0.02	0.01	0.001
8	1.860	2.306	2.896	3.355	5.041
9	1.833	2.262	2.821	3.250	4.781
10	1.812	2.228	2.764	3.169	4.587
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120	1.658	1.980	2.358	2.617	3.373
∞	1.645	1.960	2.326	2.576	3.291

Critical values of t

Alpha = .05, df = 10

Alpha = .02, df = 13

Alpha = .05, df = 120

- **Critical value of t is read across the row for the *df* in your study, to the column for your alpha.**
- **p < .05 is the most typical alpha.**
- **lower alpha** (.02 → .001, a more conservative test) **requires a higher critical value.**

Dr. David McKirnan, davidmck@uic.edu

Determining If A Result Is "Statistically Significant"

Assumptions:

- ✓ **Null hypothesis: the difference between *M*s** [or the correlation, chi square, etc.] **is > 0 or < 0 by chance alone.**
- ✓ **Statistical question: is the effect in *your* experiment different from 0 by more than chance alone?**
- ✓ **"More than chance alone" is < 5% of the time [p < .05].**

Steps:

1. **Derive the t value for the difference between groups**

$$t = \frac{(M_{group1} - M_{group2}) - 0}{\sqrt{\frac{\sum (X - M)^2_{grp1} / n - 1_{grp1}}{n_{grp1}} + \frac{\sum (X - M)^2_{grp2} / n - 1_{grp2}}{n_{grp2}}}}$$

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Steps cont.:

2. **Figure out what distribution to compare your t value to ...**
   - **Use the degrees of freedom (*df*) for this.**
   - ***df*** **= $(n_{group1} - 1) + (n_{group2} - 1)$.**
   - **The Central Limit Theorem tells us to assume there is more error (a more "flat" distribution) as *df* go down.**

4. **Use the usual criteria [alpha value] for "statistical significance" of p < .05** (unless you have good reason to use another…).

5. **Find the value on the t table that corresponds to your *df*, at your *alpha*. This is the *critical value* that your t must exceed to be considered "statistically significant".**

6. **Compare your t to the critical value, using the absolute value of t.**

Testing t

	Alpha Levels				
df	0.10	0.05	0.02	0.01	0.001
8	1.860	2.306	2.896	3.355	5.041
9	1.833	2.262	2.821	3.250	4.781
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120	1.658	1.980	2.358	2.617	3.373
∞	1.645	1.960	2.326	2.576	3.291

- ✓ **Use p < .05** (unless you want to be more conservative by using a higher value).
- ✓ **Look up your df to see what sampling distribution to compare your results to.**
- ✓ **With n = 10 per group *df* = (10-1) + (10-1) = 18.**
- ✓ **Compare your t to the critical value from the table.**
- ✓ **If the absolute value of t > the critical value, your effect is statistically significant at p < .05.**

Dr. David McKirnan, davidmck@uic.edu